Name:Answer Rubric

1. (4 points) Show that $\mathrm{SL}_2(\mathbf{Z})$ is a discrete subgroup of $\mathrm{SL}_2(\mathbf{R})$. That is, for each $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z})$, find an open neighborhood $U(g)$ in $\mathrm{SL}_2(\mathbf{R})$ such that $U(g) \cap U(g') = \emptyset$ unless $g = g'$.

Solution: Let $U(\begin{pmatrix} a & c \\ c & d \end{pmatrix})$ be the intersection of the following open set with $\mathrm{SL}_2(\mathbf{R})$ (**2 points**)

$$\left\{ \begin{pmatrix} x & y \\ z & w \end{pmatrix} : \begin{array}{l} a - 1/2 \leq x \leq a + 1/2 \\ b - 1/2 \leq y \leq b + 1/2 \\ c - 1/2 \leq z \leq c + 1/2 \\ d - 1/2 \leq w \leq d + 1/2 \end{array} \right\}.$$

If $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in U(\begin{pmatrix} a & b \\ c & d \end{pmatrix})$ with $a', b', c', d' \in \mathbf{Z}$, then $a = a'$, $b = b'$, $c = c'$ and $d = d'$(**1 point**). So assume without loss of generality that $a \neq a'$. Then $(a-1/2, a+1/2) \cap (a'-1/2, a'+1/2) = \emptyset$ and so $U(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) \cap U(\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix})$ is empty unless the two matrices are equal (**1 point**).

2. (4 points) Find the hyperbolic area of the region

$$R = \{x + iy : 0 \leq x \leq 1, 1 \leq y \leq 2\}.$$

Can you find the area of R using the Gauss-Bonnet theorem?

Solution: The area is (**2 points**)

$$\int_1^2 \left[\int_0^1 \mathrm{d}x \right] \frac{\mathrm{d}y}{y^2} = \int_1^2 \frac{\mathrm{d}y}{y^2} = \left[\frac{-1}{y} \right]_1^2 = -1/2 + 1 = 1/2.$$

We can only find the area using the Gauss-Bonnet theorem if the boundary is made up of geodesics. The horizontal boundary lines are not geodesic, so we can't apply Gauss-Bonnet (**2 points**). Alternately, if we could split this region into n triangles, the area would be $n\pi - 4(\pi/2) = (n-2)\pi$, which is not a rational number. So clearly the Gauss-Bonnet theorem does not apply.